Kardar-Parisi-Zhang Scaling of the Height of the Convective Boundary Layer and Fractal Structure of Cumulus Cloud Fields

Jon D. Pelletier

Department of Geological Sciences, Snee Hall, Cornell University, Ithaca, New York 14853

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We present the cumulative frequency-area distribution of tropical cumulus clouds as observed from satellite and space shuttle images from scales of 0.1 to 1000 km. The distribution is a power-law function of area with exponent $-0.8$. We show that this result and the fractal dimension of cloud perimeters implies that the top of the convective boundary layer (CBL) is a self-affine interface with roughness exponent or Hausdorff measure $H \approx 0.4$, the same value as that of the Kardar-Parisi-Zhang equation in $2 + 1$ dimensions. In addition, we identify dynamic scaling in a time series of the local altitude of the top of the CBL as measured with FM/CW radar backscatter intensity. A possible growth model is discussed. [S0031-9007(97)02812-3]

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In a pioneering study, Lovejoy [1] computed the fractal dimension of the perimeter of rain and cloud areas from scales of 1 to 1000 kilometers to be $1.35 \pm 0.05$. Rys and Waldvogel [2] carried out the same analysis to characterize the shape of hail clouds. For scales above 3 km they obtained a fractal dimension consistent with Lovejoy’s result. At scales below 3 km, the authors found that severely convective hail storms have perimeters with the usual Euclidean dimension of 1. Cahalan and Joseph [3] and Zhu et al. [4] extended their methodology, including the calculation of cumulative frequency-size distributions of cumulus cloud fields. They found cumulative frequency-size distributions, the number of clouds greater than or equal to an area $A$, to be a power-law function of area with an exponent close to $-1$ for some cumulus cloud scenes up to spatial scales of 10 km.

Two models have been proposed to explain aspects of the fractal structure of cumulus cloud fields. Hentschel and Procaccia [5] have considered the turbulent mixing of an initially compact cloud using a theory of turbulent diffusion to explain Lovejoy’s result. Their model does not appear to favor any particular cloud size distribution. Nagel and Raschke [6] have proposed a cellular automaton model of the atmosphere as a lattice of particles subject to a buoyant uplift upon the initiation of condensation and a nearest neighbor interaction to model entrainment of fluid by a nearby updraft. They were able to match Lovejoy’s result, but only for a particular percentage of relative humidity. Both papers model cloud dynamics only after the onset of condensation. However, it is likely that the dynamics of the growth of the convective boundary layer (CBL) prior to condensation is an important factor in the scaling of cumulus cloud fields. Clouds form by growth of the convective boundary layer (CBL), a well-mixed layer above the ground overlain by a stably stratified inversion layer, across an elevation necessary for condensation. During the daytime, the top of the CBL grows in altitude as the mixed layer is heated from below with long-wavelength outgoing radiation and develops a rough, hummocky spatial structure [7]. Studies solving the equations of fluid motion have been applied to the problem of convective boundary layer growth and cumulus cloud formation (e.g., [8]) but are of too limited a range of spatial scales to address the observed scaling.

In this Letter we calculate the cumulative frequency-area distribution of cumulus clouds to compliment Lovejoy’s [1] calculation of the perimeter fractal dimension. We find that the distribution has a power-law dependence on area with exponent $-0.8$. Using relations between the perimeter fractal dimension and size distribution of domains higher than a threshold elevation and the roughness exponent for self-affine interfaces, we infer that the top of the CBL is a self-affine interface with roughness exponent or Hausdorff measure $H \approx 0.4$. The roughness exponent or Hausdorff measure $H$ of a one-dimensional transect of an interface is defined by the relationship between the variance and the length scale over which the variance is computed: $V \propto L^{2H}$. An interface whose variance can be characterized in this way is said to be self-affine. We present a model of the growth of the CBL which is described by the Kardar-Parisi-Zhang (KPZ) [9] equation, well known from the literature on growing interfaces, which has a roughness exponent of $H = 0.4$ for $2 + 1$ dimensions [10]. The KPZ equation in $2 + 1$ dimensions describes the local height $h(x, y, t)$ of an interface in space and time:

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x, y, t),$$  

(1)

where $\eta(x, y, t)$ is Gaussian white noise with a positive mean value to model the mean positive growth of the height of the top of the CBL.

We first describe the calculation of the cumulative frequency-size distribution of cloud sizes as inferred from satellite and space shuttle photographs. We obtained global composite images from the GOES satellites prepared at the Space Science and Engineering Center at the
University of Wisconsin, Madison for five days each in
the months of October, 1995 and January, 1996. The
days were each separated by at least three days to en-
sure that each scene was largely uncorrelated. We an-
alyzed cloud images only within 30 degrees latitude of the
equator. Tropical clouds are ideal for study since
they form in environments which are nearly uniform horizon-
tally [11]. We divided each global scene into $60^\circ \times 60^\circ$
scenes centered on South America, Africa, and the Western
Pacific Ocean (regions of consistent large-scale cloud
cover). To analyze smaller scales, we obtained images of
the Earth photographed from the space shuttle. We ana-
alyzed 16 STS-67 images that satisfied the following cri-
teria: (1) considerable cumulus cloud cover untainted by
other types of clouds, (2) adequate brightness contrast
to define cloud shapes easily, (3) clouds not conspicuously
correlated with topography, and (4) clouds photographed
at a small look angle. Otherwise, the choice of the images
was random. The resolution cell size of each image type
was determined through calibration with respect to a rec-
nizable geographic shape. The resolution cell sizes of
the GOES composite and shuttle images were estimated
to be 8100 and 0.084 km$^2$, respectively. We converted
each image to a binary black and white image by mak-
ing all pixels darker than a certain threshold black and all
those lighter than the threshold white. All white areas are
defined as clouds for our analysis. The observed distrib-
ution was found to be independent of the value of the
threshold. The cumulative frequency-size distribution of
each image was computed and averaged with other images
of its type (GOES or space shuttle) at equal cloud num-
bers. Least-squares linear fits to the logarithms of the data
averaged in logarithmically spaced bins (so that the data
were uniformly weighted in log space) yielded power-law
exponents $-0.72$ and $-0.82$ for the GOES global com-
posite and space shuttle images, respectively. In order to
compare the distributions of the two types of images, the
cloud numbers for the space shuttle images were multi-
plied by a correction factor, discussed by Lovejoy [1], of
the ratio of the GOES to the space shuttle resolution cell
size to the 0.8 power. The average cumulative frequency-
size distribution for the space shuttle images scaled in this
way is plotted with the average distribution of the GOES
images in Fig. 1 along with the power-law relationship
$N(>A) \propto A^{-0.8}$. The Nagel-Raschke model predicts a
cumulative frequency-size power-law exponent of $-1.2$ [6],
consistent with the results presented in Fig. 1.

We now show that the size distribution and perimeter
fractal dimension of clouds implies that the top of
the CBL is a self-affine interface that has the same
roughness exponent as the interface produced by the
KPZ equation in $2 + 1$ dimensions. The KPZ equation
in $2 + 1$ dimensions has been solved numerically by
Amar and Family [10]. The solution is an interface
with roughness exponent $H = 0.4$. Kondev and Henley
[12] have obtained the relationship between the fractal
dimension of a contour loop of a Gaussian interface, $D$, and
the roughness exponent of the corresponding interface
as $D = 1.5 - \frac{H}{2}$. A contour loop is a connected subset
of an interface with equal elevation. Since clouds form
where the top of the CBL penetrates a threshold elevation
above which condensation begins, their base perimeters,
observable from satellite images, may be associated with
the contour loops of Kondev and Henley. Their relation,
together with the roughness exponent $H = 0.4$, predicts a
cloud perimeter fractal dimension of 1.3, consistent with
the value $1.35 \pm 0.05$ observed by Lovejoy [1] and Rys
and Waldvogel [2].

In addition, Kondev and Henley [12] have given the
size distribution of contour lengths (the probability that a
randomly chosen contour loop has a length $s$) as $N(s) \propto s^{-\tau}$, where $\tau = 1 + \frac{2 - H}{D}$. The cumulative distribution
(the number of contours with length greater than $s$) is the integral of the noncumulative distribution, $N(>s) \propto s^{-\frac{2 - H}{\tau}}$. Since the length of a contour is related
to the area it encloses by $s \propto A^{\frac{2}{D}}$ (by definition), the
cumulative distribution of areas enclosed by contours is
$N(>A) \propto A^{-\frac{2 - H}{\tau}}$. For the KPZ roughness exponent of
$H = 0.4$ this gives $N(>A) \propto A^{-0.8}$, consistent with the
size distribution of Fig. 1. Since the relationship between
the roughness exponent of the interface is related to the
observed exponents through a one-to-one function, the
observed exponents imply that the roughness exponent of
the top of the CBL is $H = 0.4$.

The altitude of the top of the CBL has been measured
using FM/CW backscatter intensity radar techniques
above a fixed position on the ground by Rowland and
Arnold [13] during a 1 h period. This time series is
shown in Fig. 2. The time series was inferred from the
radar image of Rowland and Arnold [13] by scanning the

![FIG. 1. Average cumulative frequency-size distribution, the
number of clouds greater than or equal to and area $A$, of GOES
global composite and (appropriately scaled) space shuttle cloud
images. The distribution is consistent with the KPZ model prediction $N(>A) \propto A^{-0.8}$.](https://example.com/fig1.png)
FIG. 2. Time series of the local altitude of the top of the convective boundary layer inferred from radar images of Rowland and Arnold [13].

image and computing the average height of the bright continuous region defining the transition layer between the mixed layer and the inversion layer in their image for each point in time. The time series appears to show that the top of the CBL has a constant trend of increasing altitude upon which are superimposed large fluctuations in height caused by the forcing of convective updrafts and downdrafts in the mixed layer. To characterize these dynamic fluctuations, we have performed spectral analysis on this time series after detrending the data by subtracting the least-squares linear fit. The power spectrum of this detrended series was computed using fast Fourier transform routines of Press [14]. The result is shown in Fig. 3. The power spectrum has a power-law dependence on frequency with exponent close to $-2$ for high frequencies: $S(f) \propto f^{-2}$. This result indicates that the time series of local fluctuations in height of the top of the CBL is self-affine with roughness exponent $H = 0.5$. The detrending of the data makes it impossible to determine the correct power spectral density at the lowest frequencies. This is because a self-affine time series with an exponent greater than 1 is nonstationary [15] and may therefore include a trend in the data at the lowest frequency due to the self-affine dynamics in addition to a constant trend that may be present in the process. It is impossible to separate these trends. As a result, the power spectral density of the self-affine portion of the process is indeterminable for the lowest frequencies. An uncertainty of $\pm 0.3$ was estimated for the exponent of the power spectrum by dividing the time series into four segments of equal length and computing the least-squares power spectrum exponents from each of the segments. 0.3 was the standard deviation of these exponents. The time series of the local altitude of a point on an interface described by the KPZ equation is also self-affine. The dynamic roughness exponent has been determined from simulations to be $\frac{1}{2}$ [16], which implies that the power spectrum has a power-law dependence on frequency with exponent $-\frac{3}{2}$ through the relation $\beta = 2H + 1$. The power spectrum computed in Fig. 3 also has a power-law dependence on frequency with an exponent of $-2 \pm 0.3$. The observation is consistent with the prediction, but more data would be helpful to further test the dynamic scaling with the KPZ model prediction.

We now discuss a possible model for the observations presented in this Letter. The top of the CBL develops a hummocky structure as a result of penetrative convection from below causing updrafts and downdrafts. Warner [17] has obtained measurements of the vertical velocity in clouds over a period of time and found them to be given by a Gaussian distribution and uncorrelated above a time scale small compared to the time required for cloud formation. Thus the effects of differential local penetrative convection on the height of the top of the CBL can be expressed as

$$\frac{\partial h}{\partial t} = \eta(x, y, t),$$

where $\eta(x, y, t)$ is a Gaussian white noise with positive mean value. This parametrization of the displacement of the top of the convective boundary layer is consistent with turbulent diffusion in a stably stratified atmosphere. It is well known that transport in the stably stratified atmosphere adjacent to the top of the convective boundary layer is dominated by small eddies and is governed by the diffusion equation [7]. The displacement of the top of the convective boundary layer is then directly analogous to the displacement of a Brownian particle responsible for molecular diffusion which can be modeled, at long times, by Eq. (2).

A local updraft is felt in regions nearby to where the convection penetrates the top of the CBL as a
result of viscous shear forces. Thus, viscous shear tends to smooth the interface roughness produced by differential penetrative convection. The smoothing effects of viscous shear result in a diffusive term for the interface height because this is the only term for which the effects of viscous shear will be independent of time, elevation, horizontal position, and angular orientation of the interface [18], as they should be. With the inclusion of the effect of viscous shear the equation for the interface is

\[
\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta(x, y, t). \tag{3}
\]

In addition to the effects of penetrative convection and viscous shear, the pressure gradient with height causes ascending (descending) air to expand (contract). The simplest model of this expansion and contraction is a constant growth of the interface directed everywhere perpendicular to the interface with a nonzero upward component for the layer as a whole denoted by \( \lambda \). This model corresponds to a constant pressure difference between the ascending air and the air above it. The local vertical component of growth is equal to \( \lambda [1 + (\nabla h)^2] \). If we assume that the gradients of the interface are small, or if we compare our model to only large-scale structure, we can approximate this expression as \( \lambda + \frac{\lambda}{2} (\nabla h)^2 \). This Taylor expansion procedure is the same formulation employed by Kardar, Parisi, and Zhang [9] to motivate the nonlinear term \( (\nabla h)^2 \) to model lateral growth on atomic surfaces. The resulting differential equation for the height of the interface is

\[
\frac{\partial h}{\partial t} = \nu \nabla^2 h + \lambda + \frac{\lambda}{2} (\nabla h)^2 + \eta(x, y, t). \tag{4}
\]

This is the KPZ equation. Thus, a simple model motivated from observations and symmetry principles of several key features of the growth of the CBL is described by the KPZ equation in \( 2 + 1 \) dimensions.

In conclusion, we have shown (1) that the cumulative frequency-area distribution of tropical cumulus clouds is a power-law function of area with exponent \(-0.8\), (2) that the cumulative frequency-area distribution of tropical cumulus clouds combined with the fractal dimension of their perimeters as measured by Lovejoy [1] implies that the top of the convective boundary layer is a self-affine interface with roughness exponent consistent with that of the KPZ equation in \( 2 + 1 \) dimensions, (3) the height of the top of the CBL dynamic self-affinity with a roughness exponent consistent with the prediction of the KPZ model, and (4) a simplified model of the growth of the CBL is described by the KPZ equation.