Kardar-Parisi-Zhang Scaling of the Height of the Convective Boundary Layer and Fractal Structure of Cumulus Cloud Fields

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We present the cumulative frequency-area distribution of tropical cumulus clouds as observed from satellite and space shuttle images from scales of 0.1 to 1000 km. The distribution is a power-law function of area with exponent -0.8. We show that this result and the fractal dimension of cloud perimeters implies that the top of the convective boundary layer (CBL) is a self-affine interface with roughness exponent or Hausdorff measure $H \approx 0.4$, the same value as that of the Kardar-Parisi-Zhang equation in $2 + 1$ dimensions. In addition, we identify dynamic scaling in a time series of the local altitude of the top of the CBL as measured with FM/CW radar backscatter intensity. A possible growth model is discussed. [S0031-9007(97)02812-3]

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In a pioneering study, Lovejoy [1] computed the fractal dimension of the perimeter of rain and cloud areas from scales of 1 to 1000 kilometers to be 1.35 ± 0.05. Rys and Waldvogel [2] carried out the same analysis to characterize the shape of hail clouds. For scales above 3 km they obtained a fractal dimension consistent with Lovejoy's result. At scales below 3 km, the authors found that severely convective hail storms have perimeters with the usual Euclidean dimension of 1. Cahalan and Joseph [3] and Zhu *et al.* [4] extended their methodology, including the calculation of cumulative frequency-size distributions of cumulus cloud fields. They found cumulative frequency-size distributions, the number of clouds greater than or equal to an area A, to be a power-law function of area with an exponent close to -1 for some cumulus cloud scenes up to spatial scales of 10 km.

Two models have been proposed to explain aspects of the fractal structure of cumulus cloud fields. Hentschel and Procaccia [5] have considered the turbulent mixing of an initially compact cloud using a theory of turbulent diffusion to explain Lovejoy's result. Their model does not appear to favor any particular cloud size distribution. Nagel and Raschke [6] have proposed a cellular automaton model of the atmosphere as a lattice of particles subject to a bouyant uplift upon the initiation of condensation and a nearest neighbor interaction to model entrainment of fluid by a nearby updraft. They were able to match Lovejoy's result, but only for a particular percentage of relative humidity. Both papers model cloud dynamics only after the onset of condensation. However, it is likely that the dynamics of the growth of the convective boundary layer (CBL) prior to condensation is an important factor in the scaling of cumulus cloud fields. Clouds form by growth of the convective boundary layer (CBL) , a well-mixed layer above the ground overlain by a stably stratified inversion layer, across an elevation necessary for condensation. During the daytime, the top of the CBL grows in altitude as the mixed layer is heated from below with long-wavelength outgoing radiation and develops a rough, hummocky spatial structure [7]. Studies solving the equations of fluid motion have been applied to the problem of convective boundary layer growth and cumulus cloud formation (e.g., [8]) but are of too limited a range of spatial scales to address the observed scaling.

In this Letter we calculate the cumulative frequency-area distribution of cumulus clouds to compliment Lovejoy's [1] calculation of the perimeter fractal dimension. We find that the distribution has a power-law dependence on area with exponent -0.8. Using relations between the perimeter fractal dimension and size distribution of domains higher than a threshold elevation and the roughness exponent for self-affine interfaces, we infer that the top of the CBL is a self-affine interface with roughness exponent or Hausdorff measure $H \approx 0.4$. The roughness exponent or Hausdorff measure H of a one-dimensional transect of an interface is defined by the relationship between the variance and the length scale over which the variance is computed: $V \propto L^{2H}$. An interface whose variance can be characterized in this way is said to be self-affine. We present a model of the growth of the CBL which is described by the Kardar-Parisi-Zhang (KPZ) [9] equation, well known from the literature on growing interfaces, which has a roughness exponent of $H \approx 0.4$ for $2 + 1$ dimensions [10]. The KPZ equation in $2 + 1$ dimensions describes the local height $h(x, y, t)$ of an interface in space and time:

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(x, y, t), \qquad (1)$$

where $\eta(x, y, t)$ is Gaussian white noise with a positive mean value to model the mean positive growth of the height of the top of the CBL.

We first describe the calculation of the cumulative frequency-size distribution of cloud sizes as inferred from satellite and space shuttle photographs. We obtained global composite images from the GOES satellites prepared at the Space Science and Engineering Center at the

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University of Wisconsin, Madison for five days each in the months of October, 1995 and January, 1996. The days were each separated by at least three days to ensure that each scene was largely uncorrelated. We analyzed cloud images only within 30 degrees latitude of the equator. Tropical clouds are ideal for study since they form in environments which are nearly uniform horizontally [11]. We divided each global scene into $60° \times 60°$ scenes centered on South America, Africa, and the Western Pacific Ocean (regions of consistent large-scale cloud cover). To analyze smaller scales, we obtained images of the Earth photographed from the space shuttle. We analyzed 16 STS-67 images that satisfied the following criteria: (1) considerable cumulus cloud cover untainted by other types of clouds, (2) adequate brightness contrast to define cloud shapes easily, (3) clouds not conspicuously correlated with topography, and (4) clouds photographed at a small look angle. Otherwise, the choice of the images was random. The resolution cell size of each image type was determined through calibration with respect to a recognizable geographic shape. The resolution cell sizes of the GOES composite and shuttle images were estimated to be 8100 and 0.084 km^2, respectively. We converted each image to a binary black and white image by making all pixels darker than a certain threshold black and all those lighter than the threshold white. All white areas are defined as clouds for our analysis. The observed distribution was found to be independent of the value of the threshold. The cumulative frequency-size distribution of each image was computed and averaged with other images of its type (GOES or space shuttle) at equal cloud numbers. Least-squares linear fits to the logarithms of the data averaged in logarithmically spaced bins (so that the data were uniformly weighted in log space) yielded power-law exponents -0.72 and -0.82 for the GOES global composite and space shuttle images, respectively. In order to compare the distributions of the two types of images, the cloud numbers for the space shuttle images were multiplied by a correction factor, discussed by Lovejoy [1], of the ratio of the GOES to the space shuttle resolution cell size to the 0.8 power. The average cumulative frequency-size distribution for the space shuttle images scaled in this way is plotted with the average distribution of the GOES images in Fig. 1 along with the power-law relationship $N(>A) \propto A^{-0.8}$. The Nagel-Raschke model predicts a cumulative frequency-size power-law exponent of -1.2 [6], inconsistent with the results presented in Fig. 1.

We now show that the size distribution and perimeter fractal dimension of clouds implies that the top of the CBL is a self-affine interface that has the same roughness exponent as the interface produced by the KPZ equation in 2 + 1 dimensions. The KPZ equation in 2 + 1 dimensions has been solved numerically by Amar and Family [10]. The solution is an interface with roughness exponent $H \approx 0.4$. Kondev and Henley [12] have obtained the relationship between the fractal dimension of a contour loop of a Gaussian interface, D, and the roughness exponent of the corresponding interface as $D = 1.5 - \frac{H}{2}$. A contour loop is a connected subset of an interface with equal elevation. Since clouds form where the top of the CBL penetrates a threshold elevation above which condensation begins, their base perimeters, observable from satellite images, may be associated with the contour loops of Kondev and Henley. Their relation, together with the roughness exponent $H \approx 0.4$, predicts a cloud perimeter fractal dimension of 1.3, consistent with the value 1.35 ± 0.05 observed by Lovejoy [1] and Rys and Waldvogel [2].

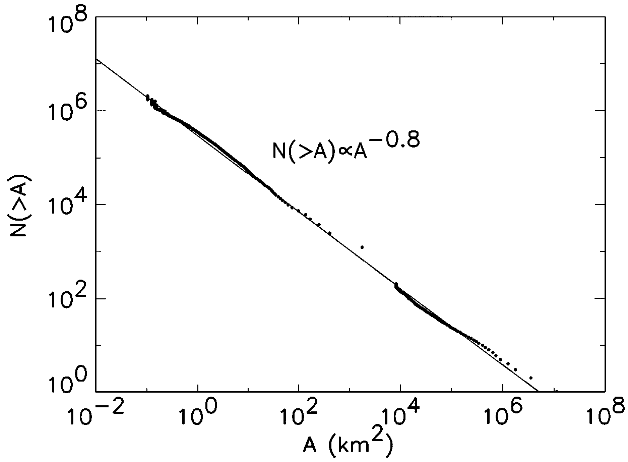

FIG. 1. Average cumulative frequency-size distribution, the number of clouds greater than or equal to and area A, of GOES global composite and (appropriately scaled) space shuttle cloud images. The distribution is consistent with the KPZ model prediction $N(>A) \propto A^{-0.8}$.

In addition, Kondev and Henley [12] have given the size distribution of contour lengths (the probability that a randomly chosen contour loop has a length s) as $N(s) \propto s^{-\tau}$, where $\tau = 1 + \frac{2-H}{D}$. The cumulative distribution (the number of contours with length greater than s) is the integral of the noncumulative distribution, $N(> s) \propto s^{-\frac{2-H}{D}}$. Since the length of a contour is related to the area it encloses by $s \propto A^{\frac{D}{2}}$ (by definition), the cumulative distribution of areas enclosed by contours is $N(>A) \propto A^{-\frac{2-H}{2}}$. For the KPZ roughness exponent of $H \approx 0.4$ this gives $N(>A) \propto A^{-0.8}$, consistent with the size distribtion of Fig. 1. Since the relationship between the roughness exponent of the interface is related to the observed exponents through a one-to-one function, the observed exponents imply that the roughness exponent of the top of the CBL is $H \approx 0.4$.

The altitude of the top of the CBL has been measured using FM/CW backscatter intensity radar techniques above a fixed position on the ground by Rowland and Arnold [13] during a 1 h period. This time series is shown in Fig. 2. The time series was inferred from the radar image of Rowland and Arnold [13] by scanning the

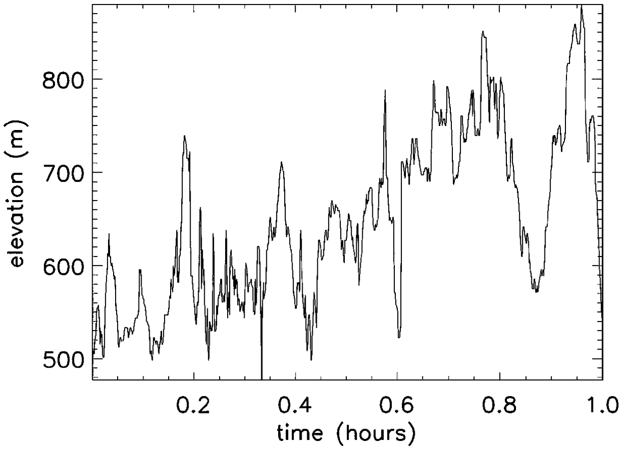

FIG. 2. Time series of the local altitude of the top of the convective boundary layer inferred from radar images of Rowland and Arnold [13].

image and computing the average height of the bright continuous region defining the transition layer between the mixed layer and the inversion layer in their image for each point in time. The time series appears to show that the top of the CBL has a constant trend of increasing altitude upon which are superimposed large fluctuations in height caused by the forcing of convective updrafts and downdrafts in the mixed layer. To characterize these dynamic fluctuations, we have performed spectral analysis on this time series after detrending the data by subtracting the least-squares linear fit. The power spectrum of this detrended series was computed using fast Fourier transform routines of Press [14]. The result is shown in Fig. 3. The power spectrum has a power-law dependence on frequency with exponent close to -2 for high frequencies: $S(f) \propto f^{-2}$. This result indicates that the time series of local fluctuations in height of the top of the CBL is self-affine with roughness exponent $H \approx 0.5$. The detrending of the data makes it impossible to determine the correct power spectral density at the lowest frequencies. This is because a self-affine time series with an exponent greater than 1 is nonstationary [15] and may therefore include a trend in the data at the lowest frequency due to the self-affine dynamics in addition to a constant trend that may be present in the process. It is impossible to separate these trends. As a result, the power spectral density of the self-affine portion of the process is indeterminable for the lowest frequencies. An uncertainty of ± 0.3 was estimated for the exponent of the power spectrum by dividing the time series into four segments of equal length and computing the least-squares power spectrum exponents from each of the segments. 0.3 was the standard deviation of these exponents. The time series of the local altitude of a point on an interface described by the KPZ equation is also self-affine. The dynamic roughness exponent has been determined from simulations to be $\frac{1}{4}$ [16], which implies that the power spectrum has a power-law dependence on frequency with exponent $-\frac{3}{2}$ through the relation $\beta = 2H + 1$. The power spectrum computed in Fig. 3 also has a power-law dependence on frequency with an exponent of -2 ± 0.3. The observation is consistent with the prediction, but more data would be helpful to further test the dynamic scaling with the KPZ model prediction.

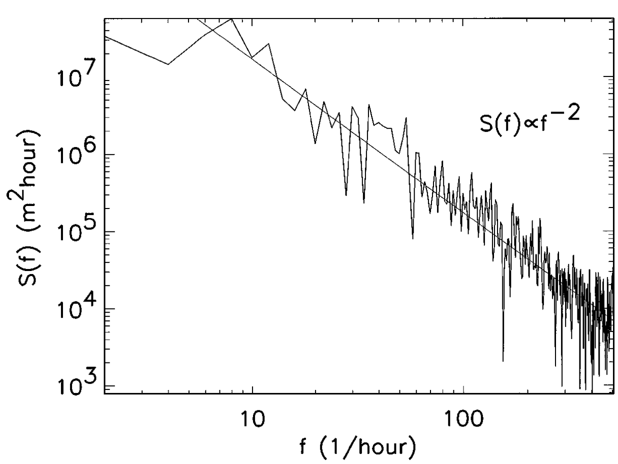

FIG. 3. Power spectrum of the time series of Fig. 2. The time series is scale invariant with an exponent of approximately -2. This differs from the KPZ model prediction of -1.5, but there is a large uncertainty in the exponent due to limited data length as described in the text.

We now discuss a possible model for the observations presented in this Letter. The top of the CBL develops a hummocky structure as a result of penetrative convection from below causing updrafts and downdrafts. Warner [17] has obtained measurements of the vertical velocity in clouds over a period of time and found them to be given by a Gaussian distribution and uncorrelated above a time scale small compared to the time required for cloud formation. Thus the effects of differential local penetrative convection on the height of the top of the CBL can be expressed as

$$\frac{\partial h}{\partial t} = \eta(x, y, t), \tag{2}$$

where $\eta(x, y, t)$ is a Gaussian white noise with positive mean value. This parametrization of the displacement of the top of the convective boundary layer is consistent with turbulent diffusion in a stably stratified atmosphere. It is well known that transport in the stably stratified atmosphere adjacent to the top of the convective boundary layer is dominated by small eddies and is governed by the diffusion equation [7]. The displacement of the top of the convective boundary layer is then directly analogous to the displacement of a Brownian particle responsible for molecular diffusion which can be modeled, at long times, by Eq. (2).

A local updraft is felt in regions nearby to where the convection penetrates the top of the CBL as a result of viscous shear forces. Thus, viscous shear tends to smooth the interface roughness produced by differential penetrative convection. The smoothing effects of viscous shear result in a diffusive term for the interface height because this is the only term for which the effects of viscous shear will be independent of time, elevation, horizontal position, and angular orientation of the interface [18], as they should be. With the inclusion of the effect of viscous shear the equation for the interface is

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta(x,y,t). \tag{3}$$

In addition to the effects of penetrative convection and viscous shear, the pressure gradient with height causes ascending (descending) air to expand (contract). The simplest model of this expansion and contraction is a constant growth of the interface directed everywhere perpendicular to the interface with a nonzero upward component for the layer as a whole denoted by λ. This model corresponds to a constant pressure difference between the ascending air and the air above it. The local vertical component of growth is equal to $\lambda[1 + (\nabla h)^2]^{\frac{1}{2}}$. If we assume that the gradients of the interface are small, or if we compare our model to only large-scale structure, we can approximate this expression as $\lambda + \frac{\lambda}{2}(\nabla h)^2$. This Taylor expansion procedure is the same formulation employed by Kardar, Parisi, and Zhang [9] to motivate the nonlinear term $(\nabla h)^2$ to model lateral growth on atomic surfaces. The resulting differential equation for the height of the interface is

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \lambda + \frac{\lambda}{2}(\nabla h)^2 + \eta(x,y,t). \tag{4}$$

This is the KPZ equation. Thus, a simple model motivated from observations and symmetry principles of several key features of the growth of the CBL is described by the KPZ equation in $2 + 1$ dimensions.

In conclusion, we have shown (1) that the cumulative frequency-area distribution of tropical cumulus clouds is a power-law function of area with exponent -0.8, (2) that the cumulative frequency-area distribution of tropical cumulus clouds combined with the fractal dimension of their perimeters as measured by Lovejoy [1] implies that the top of the convective boundary layer is a self-affine interface with roughness exponent consistent with that of the KPZ equation in $2 + 1$ dimensions, (3) the height of the top of the CBL dynamic self-affinity with a roughness exponent consistent with the prediction of the KPZ model, and (4) a simplified model of the growth of the CBL is described by the KPZ equation.

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